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$\frac{3}{8}$	$\left\{\frac{a}{39}(49 \pm 4\sqrt{55}) = 2.017a \text{ or } 0.496a\right.$
$\frac{9}{16}$	$\left\{\frac{a}{384}(461 \pm 13\sqrt{385}) = 1.865a \text{ or } 0.536a\right.$
$\frac{3}{4}$	$\left\{\begin{array}{ll}\frac{2}{3}a & \text{Low Point } (b = \frac{2}{3}a) \\ \frac{3}{2}a & (b = a)\end{array}\right.$
$\frac{13}{16}$	$\left\{\frac{a}{128}(133 \pm \sqrt{1305}) = 1.221a \text{ or } 0.757a\right.$
$\frac{7}{8}$	$a \qquad (b = \frac{1}{2}a)$

CALCULUS.

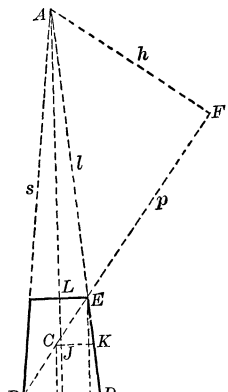
407. Proposed by PAUL CAPRON, Annapolis, Maryland.

A coffee pot in the form of a conical frustum, 10 inches high, with a lower base 8 inches in diameter and an upper base 6 inches in diameter, is held on a slant so that the lower base is barely covered by the coffee within, and the upper base is barely uncovered. How much coffee does the pot contain?

I. Solution by H. S. Uhler, Yale University.

The following solution may be of interest because it is based exclusively on theorems of elementary geometry. The diagram (drawn to scale) represents a plane section passing through the axis $\overline{IA}$ of the frustum $BDEL$ and the major axis $\overline{BE}$ of the elliptical free-surface of the coffee. The required volume will be gotten by subtracting the volume of the oblique cone ABE from that of the right cone ABD. We are given that $\overline{IL} = 10$ in., $\overline{ID} = 4$ in., and $\overline{LE} = 3$ in.

From the similar triangles AID and ALE, $\overline{AI} : \overline{ID} = \overline{AL} : \overline{LE}$ or $\overline{AI} : 4 = (\overline{AI} - 10) : 3$, hence $\overline{AI} = 40$ in. Hence, the volume of the right cone equals $(\frac{640}{3})\pi$ cu. in.



The altitude of the oblique cone may be obtained from the right triangles ABF and AEF, for, $s^2 = (2a + p)^2 + h^2$ and $l^2 = p^2 + h^2$, where a denotes the semi-major axis of the ellipse. That $a = 0.5\sqrt{149}$ may be seen at once from the right triangle BHE since $\overline{BH} = 7$ and $\overline{HE} = 10$. $s^2 = (\overline{BI})^2 + (\overline{AI})^2 = 1616$ and $l^2 = (\overline{LE})^2 + (\overline{LA})^2 = 909$. Elimination of p from the twoliteral equations gives $h = 240/\sqrt{149}$ in. It remains to find the semi-minor axis b of the ellipse. This axis will pass through C, the middle point of $\overline{BE}$, it will be perpendicular to the plane of the diagram, and it will be a chord of the circle whose radius is $\overline{JK}$ and whose plane is parallel to the bases of the given frustum. Since C bisects $\overline{BE}$ and $\overline{CK}$ is parallel to $\overline{BD}$, $\overline{BG} = 0.5\overline{BH} = 3.5$. $\overline{CJ} = \overline{BI} - \overline{BG} = 4 - 3.5 = 0.5$. $\overline{JK} = 0.5(\overline{ID} + \overline{LE}) = 3.5$. Since the minor axis of the ellipse constitutes the chord of a circle of radius 3.5 in. and is at a distance 0.5 in. from the center it follows that $b^2 = (3.5 - 0.5)(3.5 + 0.5) = 12$. Hence, $b = 2\sqrt{3}$ in. The area of the ellipse $= \pi ab = \pi\sqrt{3 \times 149}$;hence, the volume of the oblique cone equals

$$\tfrac{1}{3}(\pi\sqrt{3 \times 149})\left(\frac{240}{\sqrt{149}}\right) = 80\pi\sqrt{3} \text{ cu. in.}$$

Consequently, the volume of the coffee equals

$$\frac{80\pi}{3}(8 - 3\sqrt{3}) \text{ cu. in.} \doteqdot 234.894,585,349,6 \text{ cu. in.}$$

II. Solution by H. S. Uhler, Yale University.

Let us choose a set of rectangular coördinate axes such that the origin is at the center of the lower base of the frustum, the axis of y lies in the plane of the lower base with its negative segment passing through the lower end of the major axis of the elliptical free-surface of the coffee, and the axis of z is directed from the origin toward the apex of the completed cone. If the axes of y and z are drawn to the right and upward respectively, the axis of x will be out from the plane of the paper toward the reader.

We shall take as element of volume the frustum of an oblique pyramid bounded as follows: Two of the lateral faces are portions of the surfaces of right circular cones having the same basal plane and apex as the given cone, and with radii r and $r + dr$. The two remaining lateral faces lie in planes which contain the axis of the given cone (*i. e.*, OZ) and make angles θ and $\theta + d\theta$ with the plane YOZ. The upper base of this frustum is parallel to the lower base and passes through the point of intersection of the plane of the ellipse with the edge which is nearest to the axis OZ and to the plane YOZ. No finite error can arise from neglecting the little solid between the plane of the ellipse and the plane of the upper base of the chosen element of volume because it is an infinitesimal of the third order, whereas the element of volume just defined is of the second order.

The volume of the element may be looked upon as the difference between the volumes of two pyramids. The area of the base of the larger pyramid equals $rdrd\theta$. The altitude of this pyramid is identical with that of the given cone, and is obviously equal to 40 inches. Hence, the volume of the taller pyramid equals $40rdrd\theta/3$. If h denotes the altitude of the element of volume then the altitude of the shorter pyramid will be $40 - h$. Since the volumes of two similar pyramids are to each other as the cubes of their altitudes it follows that the volume of the lesser pyramid equals $(40 - h)^3 rdrd\theta/4800$. Consequently, the element of volume equals $40rdrd\theta/3 - (40 - h)^3 \cdot rdrd\theta/4800$.

h may be obtained from its definition as follows. The plane of the ellipse contains the line $y = -4$ and the point $(0, 3, 10)$ so that its equation is $-10y + 7z = 40$. The direction cosines, with reference to the axes of y and z respectively, of the edge drawn from $(r \sin\theta, r\cos\theta, 0)$ to $(0, 0, 40)$ are easily found to be in the ratio $-r\cos\theta : 40$, so that one equation of the edge is

$$-\frac{y}{r\cos\theta} = \frac{z - 40}{40}.$$

Elimination of y between this equation and that of the plane of the ellipse gives

$$z = h = \frac{40(4 + r\cos\theta)}{28 + r\cos\theta}.$$

Finally, the element of volume reduces to

$$\frac{40}{3} rdrd\theta - \frac{184{,}320 rdrd\theta}{(28 + r\cos\theta)^3}.$$

Volume of coffee $\equiv V = \frac{80}{3}\int_0^{\pi} d\theta \int_0^{4} rdr - 368{,}640 \int_0^{\pi} d\theta \int_0^{4} \frac{rdr}{(28 + r\cos\theta)^3}$. Making use of the fact that

$$\int \frac{xdx}{(a + bx)^3} = \frac{1}{b^2}\left[-\frac{1}{a + bx} + \frac{a}{2(a + bx)^2}\right]$$

we find

$$V = \frac{640\pi}{3} - 92{,}160 \int_0^{\pi} \left[-\frac{1}{7 + \cos\theta} + \frac{7}{2(7 + \cos\theta)^2} + \frac{1}{14}\right] \sec^2\theta d\theta$$

or

$$V = \frac{640\pi}{3} - \frac{46{,}080}{7} \int_0^{\pi} \frac{d\theta}{(7 + \cos\theta)^2}.$$

The tabular integrals

$$\int \frac{dx}{(a + b\cos x)^2} = \frac{1}{a^2 - b^2}\left[-\frac{b\sin x}{a + b\cos x} + a\int \frac{dx}{a + b\cos x}\right]$$

$$\int \frac{dx}{a + b\cos x} = \frac{-1}{\sqrt{a^2 - b^2}} \cdot \sin^{-1}\left[\frac{b + a\cos x}{a + b\cos x}\right]$$

lead to

$$\int_0^{\pi} \frac{d\theta}{(7+\cos\theta)^2} = \frac{7\sqrt{3}}{576}\pi$$

so that

$$V = \frac{80\pi}{3}(8 - 3\sqrt{3}) \text{ cu. in.}$$

$$V \doteqdot 234.894{,}585{,}349{,}6 \text{ cu. in.}$$

Also solved by G. A. KNAPP, C. N. SCHMALL, GEORGE PAASWELL, H. N. CARLETON, O. S. ADAMS, and the PROPOSER.

408. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

The ellipse $(x^2/81) + (y^2/16) = 1$ is revolved around the y-axis. Find the area of the surface generated.

SOLUTION BY CLYDE S. ATCHISON, Washington and Jefferson College.

From the given equation, we have

$$x = \frac{9}{4}\sqrt{16 - y^2}; \quad dx = -\frac{9y\,dy}{4\sqrt{16 - y^2}}; \quad \text{and} \quad ds = \frac{1}{4}\sqrt{\frac{256 + 65y^2}{16 - y^2}} \cdot dy.$$

Then the area of the surface generated is

$$2\pi \int_{y=-4}^{y=4} x \cdot ds = 2\pi \int_{-4}^{4} \left(\frac{9}{4}\sqrt{16 - y^2}\right)\left(\frac{1}{4}\sqrt{\frac{256 + 65y^2}{16 - y^2}}\right) dy = \frac{9\pi}{8}\int_{-4}^{4} \sqrt{256 + 65y^2} \cdot dy$$

$$= \frac{9\pi\sqrt{65}}{16}\left[y \cdot \sqrt{y^2 + \frac{256}{65}} + \frac{256}{65}\log\left(y + \sqrt{y^2 + \frac{256}{65}}\right)\right]_{-4}^{4}$$

$$= \frac{9\pi\sqrt{65}}{16}\left\{ \frac{144}{\sqrt{65}} + \frac{256}{65}\log\left(4 + \frac{36}{\sqrt{65}}\right) + \frac{144}{\sqrt{65}} - \frac{256}{65}\log\left(-4 + \frac{36}{\sqrt{65}}\right)\right\},$$

$$= 162\pi + \frac{144\pi}{\sqrt{65}}\log\frac{9 + \sqrt{65}}{9 - \sqrt{65}}.$$

Also solved by A. M. HARDING, NELLIE L. INGALS, HORACE OLSON, C. C. YEN, G. W. HARTWELL, H. C. FEEMSTER, J. A. ECKSON, GEORGE PAASWELL, O. S. ADAMS, and PAUL CAPRON.

MECHANICS.

321. Proposed by E. J. MOULTON, Northwestern University.

The attraction, A, in any direction, due to a homogeneous sphere, on a particle at the center of the sphere, using the Newtonian law, is obviously zero. Find the error in the following method of computing A. Take cylindrical coördinates with origin at the center of the sphere; let the Z-axis extend in the direction of the attraction to be computed, and let r, θ be the polar coördinates used. Let δ be the density and R the radius of the sphere, and k the constant of gravitation. Then

$$A = \int_{z=-R}^{z=R}\int_{r=0}^{r=\sqrt{R^2-z^2}}\int_{\theta=0}^{\theta=2\pi} \frac{k\delta r\,dz\,d\theta\,dr}{[r^2+z^2]^{\frac{3}{2}}} \tag{1}$$

$$= 2\pi k\delta \int_{z=-R}^{z=R}\left[\frac{-z}{(r^2+z^2)^{\frac{1}{2}}}\right]_{r=0}^{z=\sqrt{R^2-z^2}} dz \tag{2}$$

$$= 2\pi k\delta \int_{-R}^{R}\left[\frac{-z}{R} + 1\right] dz \tag{3}$$

$$= 4\pi k\delta R. \tag{4}$$